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The supersymmetric Yang–Mills theory on noncommutative geometry

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Recently, we have found the supersymmetric counterpart of the spectral triple. When we restrict the representation space to the fermionic functions of matter fields, the counterpart, which we name "the triple", reduces to the original spectral triple, which defines noncommutative geometry. We see that the fluctuation of the supersymmetric Dirac operator induced by algebra in the triple generates a vector supermultiplet that mediates the gauge interaction. Following the supersymmetric version of the spectral action principle, we calculate the heat kernel expansion of the square of the fluctuated Dirac operator and obtain the correct supersymmetric Yang–Mills action with $U(N)$ gauge symmetry.

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1. Introduction

The standard model of high energy physics coupled to gravity was derived on the basis of noncommutative geometry (NCG) by Connes and his co-workers [1–3]. Their result was that, if the space-time was a product of a continuous Riemannian manifold M and a finite space F of K-theoretic (KO)-dimension 6, gauge theories of the standard model could be uniquely derived [4].

The framework of an NCG is specified by a set called a spectral triple [5]. Let it be denoted by $(\mathcal{H}_0, \mathcal{A}_0, \mathcal{D}_0)$. Here, $\mathcal{A}_0$ is a noncommutative complex algebra, acting on the Hilbert space $\mathcal{H}_0$, whose elements correspond to the spinorial wave functions of physical matter fields, while the Dirac operator $\mathcal{D}_0$ is a self-adjoint operator with compact resolvent. The operator plays the role of the inverse of the infinitesimal unit of length ds of ordinary geometry and satisfies the condition that $[\mathcal{D}_0, a]$ is bounded for arbitrary a in $\mathcal{A}_0$. The $Z/2$ grading γ and the real structure $\mathcal{J}$ are taken into account to determine the KO dimension. These axioms are given in the Euclidean signature [6].

The automorphisms of the algebra $\mathcal{A}_0$ are separated into equivalence classes under its normal subgroup. In the same way, the space of metrics, i.e. Dirac operators, has a foliation of equivalence classes and the internal fluctuation of a metric is given as follows:

$$\tilde{\mathcal{D}}_0 = \mathcal{D}_0 + A + JAJ^{-1}, \ \ A = \sum a_i[\mathcal{D}, b_i], \ \ a_i, b_i \in \mathcal{A}. \tag{1}$$

The fluctuation $A + JAJ^{-1}$ for the Dirac operator in the manifold $\mathcal{D}_{0M} = i\gamma^\mu\nabla_\mu \otimes 1$ gives the gauge vector field, while that for the fluctuation of the Dirac operator in the finite space $\mathcal{D}_{0F}$ gives the Higgs field [6,7].

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The action of the NCG model is obtained by the spectral action principle and is expressed by

$$\langle\psi\tilde{\mathcal{D}}_0\psi\rangle + \mathrm{Tr}(f(P)). \tag{2}$$

Here the first term stands for the matter action and ψ is a fermionic field that belongs to $\mathcal{H}_0$. The second term represents the bosonic part that depends only on the spectrum of the squared Dirac operator $P = \tilde{\mathcal{D}}_0^2$, and $f(x)$ is an auxiliary smooth function on a 4D compact Riemannian manifold without boundary [8]. It includes not only non-Abelian gauge theory but also Higgs field theory and Einstein's general relativity.

In our previous paper [9], we extended the spectral triple defined on the flat Riemannian manifold to a counterpart in the supersymmetric theory that may overcome various shortcomings of the standard model [10], for instance, the hierarchy problem, the fact that many free parameters are to be determined by experiments, and the lack of unification of the running gauge-coupling constants of the renormalization group. We referred to the supersymmetric counterpart as simply "the triple" and the triple on the manifold was denoted by $(\mathcal{A}_M, \mathcal{H}_M, \mathcal{D}_M)$. A component of the functional space $\mathcal{H}_M$ in our counterpart was made up of spinor and scalar wave functions of $C^\infty(M)$ that constructed a chiral or an antichiral supermultiplet. A $Z/2$ grading of the space $\mathcal{H}_M$ was given by the chirality of the supersymmetry transformation. So, the functional space in the manifold, $\mathcal{H}_M$, was a direct sum of $\mathcal{H}_+$ of chiral supermultiplets and $\mathcal{H}_-$ of antichiral supermultiplets. The algebra $\mathcal{A}_M$ was also separated to two subsets $\mathcal{A}_+$ and $\mathcal{A}_-$, each of which was represented on $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively. On the other hand, the representation of the Dirac operator $\mathcal{D}_M$ was defined on the whole $\mathcal{H}_M$. While the ingredients of NCG were constructed in the Euclidean signature, the above construction was performed in the Minkowskian signature in order to incorporate supersymmetry. We note that the triple did not define a new NCG. However, by projecting $\mathcal{H}_M$ to the fermionic part $\mathcal{H}_{0M}$ and changing the signature from Minkowskian to Euclidean by the Wick rotation, we found that it reduced to the theory constructed on the original spectral triple.

In this paper, we faithfully obey the original idea of NCG and investigate whether it is possible to introduce a vector supermultiplet that induces a gauge field through internal fluctuation of the Dirac operator $\mathcal{D}_M$. In order to incorporate gauge quantum numbers of matter particles and their superpartners and to obtain their mass terms, we also introduce the finite space F and their supersymmetric counterpart denoted by $(\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F)$. We then calculate the spectral action using the modified total Dirac operators on $M \otimes F$ following the supersymmetric version of the spectral action principle.

This paper is organized as follows. In Sect. 2, we review the construction of the supersymmetric counterpart extended from the original spectral triple on the manifold. We also introduce the counterpart on the finite space so that the triple defined on $M \otimes F$ will be given. In Sect. 3, we calculate the internal fluctuation of the Dirac operator that induces the vector supermultiplet with $U(N)$ internal degrees of freedom. We can define an adequate supersymmetric invariant product of elements in $\mathcal{H}_M$ and obtain a bilinear form similar to the first term in Eq. (2). It represents the action for the chiral and antichiral supermultiplets of matter fields and their superpartners that interact with fields of the vector supermultiplet. In Sect. 4, we calculate the square of the fluctuated Dirac operator and Seelay–DeWitt coefficients of heat kernel expansion so that we obtain the correct action of the supersymmetric Yang–Mills theory.

2. Supersymmetrically extended triple

The supersymmetric counterpart of the spectral triple on the flat Riemannian manifold M was introduced in our previous paper [9]. In this section, let us review it and introduce the counterpart

$(\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F)$ on the finite space F in order to construct the triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ on $M \otimes F$. The functional space $\mathcal{H}$ is the product denoted by

$$\mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_F. \tag{3}$$

The functional space on the Minkowskian space-time manifold is the direct sum of two subsets, $\mathcal{H}_+$ and $\mathcal{H}_-$:

$$\mathcal{H}_M = \mathcal{H}_+ \oplus \mathcal{H}_-. \tag{4}$$

The element of $\mathcal{H}_M$ is given by

$$\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = \Phi_+ + \Phi_-, \tag{5}$$

$$\Phi_+ = \begin{pmatrix} \Psi_+ \\ 0^3 \end{pmatrix} \in \mathcal{H}_+,\ \Phi_- = \begin{pmatrix} 0^3 \\ \Psi_- \end{pmatrix} \in \mathcal{H}_-. \tag{6}$$

Here, Ψ_+, Ψ_- are denoted by

$$(\Psi_+)_i = (\varphi_+(x), \psi_{+\alpha}(x), F_+(x))^T,\ i = 1, 2, 3, \tag{7}$$

and

$$(\Psi_-)_{\bar{i}} = (\varphi_-(x), \bar{\psi}_-^{\dot{\alpha}}(x), F_-(x))^T,\ \bar{i} = 1, 2, 3, \tag{8}$$

in the vector notation. Here, φ_+ and F_+ of Ψ_+ are complex scalar functions with mass dimension one and two, respectively, and $\psi_{+\alpha}$, $\alpha = 1, 2$ are the Weyl spinors on the space-time M that have mass dimension $\frac{3}{2}$ and transform as the $(\frac{1}{2}, 0)$ representation of the Lorentz group, $SL(2, C)$. The $\Psi_+(x)$ obey the following chiral supersymmetry transformation and form a chiral supermultiplet:

$$\begin{cases} \delta_\xi \varphi_+ = \sqrt{2}\xi^\alpha \psi_{+\alpha}, \\ \delta_\xi \psi_{+\alpha} = i\sqrt{2}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}\partial_\mu \varphi_+ + \sqrt{2}\xi_\alpha F_+, \\ \delta_\xi F_+ = i\sqrt{2}\bar{\xi}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu \psi_{+\alpha}. \end{cases} \tag{9}$$

On the other hand, the $\bar{\psi}^{\dot{\alpha}}$ transform as the $(0, \frac{1}{2})$ of $SL(2, C)$ and the $\Psi_-(x)$ form an antichiral supermultiplet that obeys the antichiral supersymmetry transformation as follows:

$$\begin{cases} \delta_\xi \varphi_- = \sqrt{2}\bar{\xi}_{\dot{\alpha}}\bar{\psi}_-^{\dot{\alpha}}, \\ \delta_\xi \bar{\psi}_-^{\dot{\alpha}} = i\sqrt{2}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\xi_\alpha \partial_\mu \varphi_- + \sqrt{2}\bar{\xi}^{\dot{\alpha}} F_-, \\ \delta_\xi F_- = i\sqrt{2}\xi^\alpha \sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \bar{\psi}_-^{\dot{\alpha}}. \end{cases} \tag{10}$$

The $Z/2$ grading of the functional space $\mathcal{H}_M$ is given by an operator that is defined by

$$\gamma_M = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \tag{11}$$

In this basis, we have $\gamma_M(\Psi_+) = -i$ and $\gamma_M(\Psi_-) = i$. Hereafter, we suitably abbreviate the unit matrices or subscripts that denote the sizes of unit and zero matrices.

Let us discuss the space $\mathcal{H}_F$. $\mathcal{H}_F$ is the space with the basis of the labels q_L^a and q_R^a, which correspond to some matter particles and their superpartners, such as quarks, squarks, and auxiliary fields.

Here a is the index, $a = 1, \ldots, N$, which denotes internal degrees of freedom. L and R denote the eigenstates of the $Z/2$ grading γ_F for the discrete space, which is defined by

$$\gamma_F = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{12}$$

in the basis of $\mathcal{H}_F$ given by

$$Q^a = \begin{pmatrix} q^a_L \\ q^a_R \end{pmatrix} \in \mathcal{H}_F. \tag{13}$$

In this basis, we have $\gamma_F(q^a_L) = -1$, and $\gamma_F(q^a_R) = 1$. The wave functions of the supermultiplets in $\mathcal{H}$ are expressed by $(\Psi_+, \Psi_-) \otimes (q^a_L, q^a_R)$. In order to avoid fermion doubling [11,12], we require that the physical wave functions obey the following condition:

$$\gamma = \gamma_M \gamma_F = i. \tag{14}$$

Then, for the supermultiplet, which is a set of a left-handed fermionic matter field and its superpartner and auxiliary field, we have

$$\begin{aligned} \Psi^a_L &= q^a_L \otimes \Phi_+ \\ &= q^a_L \otimes (\varphi_+, \psi_{+\alpha}, F_+, 0^3)^T, \end{aligned} \tag{15}$$

in the Minkowskian signature and the physical wave functions of the supermultiplet amount to

$$q^a_{L\alpha} = q^a_L \otimes \psi_{+\alpha}(x), \tag{16}$$

$$\tilde{q}^a_L(x) = q^a_L \otimes \varphi_+(x), \tag{17}$$

$$F^a_L(x) = q^a_L \otimes F_+(x). \tag{18}$$

For the wave functions of the right-handed fermionic matter field, we have

$$\Psi^a_R(x) = q^a_R \otimes \Phi_- = q^a_R \otimes (0^3, \varphi^*_-, \bar{\psi}^{\dot{\alpha}}_-, F^*_-)^T \tag{19}$$

and

$$q^{a\dot{\alpha}}_R(x) = q^a_R \otimes \bar{\psi}^{\dot{\alpha}}_-(x), \tag{20}$$

$$\tilde{q}^a_R(x) = q^a_R \otimes \varphi_-(x)^*, \tag{21}$$

$$F^a_R(x) = q^a_R(x) \otimes F_-(x)^*. \tag{22}$$

For the state $\Psi \in \mathcal{H}_M$, the charge conjugate state Ψ^c is given by

$$\Psi^c = \begin{pmatrix} \Psi^c_+ \\ \Psi^c_- \end{pmatrix}. \tag{23}$$

The antilinear operator $\mathcal{J}_M$ is defined by

$$\Psi^c = \mathcal{J}_M \Psi = C\Psi^*, \tag{24}$$

so that it is given by

$$\mathcal{J}_M = C \otimes *, \tag{25}$$

where C is the following charge conjugation matrix:

$$C = \left(\begin{array}{ccc|ccc} & & & 1 & 0 & 0 \\ & \mathbf{0} & & 0 & \epsilon_{\alpha\beta} & 0 \\ & & & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & & & \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} & 0 & & \mathbf{0} & \\ 0 & 0 & 1 & & & \end{array}\right), \tag{26}$$

and $*$ is the complex conjugation. The operator $\mathcal{J}_M$ obeys the following relation:

$$\mathcal{J}_M \gamma_M = \gamma_M \mathcal{J}_M. \tag{27}$$

The real structure J_M is now expressed for the basis of the Hilbert space $(\Phi, \Phi^c)^T$ in the following form:

$$J_M = \begin{pmatrix} 0 & \mathcal{J}_M^{-1} \\ \mathcal{J}_M & 0 \end{pmatrix}. \tag{28}$$

The $Z/2$ grading Γ_M on the basis is expressed by

$$\Gamma_M = \begin{pmatrix} \gamma_M & 0 \\ 0 & \gamma_M \end{pmatrix}. \tag{29}$$

In the finite space, the antilinear operator $\mathcal{J}_F$ is defined by

$$\mathcal{J}_F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes *. \tag{30}$$

Then the anti-matter particle supermultiplet Q_a^c is related to Q_a by

$$Q_a^c = \mathcal{J}_F Q^a. \tag{31}$$

On the basis $(Q_a, Q_a^c)^T$, the real structure J_F and the $Z/2$ grading are expressed as follows:

$$J_F = \begin{pmatrix} 0 & \mathcal{J}_F^{-1} \\ \mathcal{J}_F & 0 \end{pmatrix}, \; \Gamma_F = \begin{pmatrix} \gamma_F & 0 \\ 0 & \gamma_F \end{pmatrix}. \tag{32}$$

Corresponding to the construction of the functional spaces (3) and (4), the algebra $\mathcal{A}$ represented by them are expressed as

$$\mathcal{A} = \mathcal{A}_M \otimes \mathcal{A}_F, \tag{33}$$

$$\mathcal{A}_M = \mathcal{A}_+ \oplus \mathcal{A}_-. \tag{34}$$

Here an element u_a of $\mathcal{A}_+$, which acts on $\mathcal{H}_+$, and an element $\bar{u}_a$ of $\mathcal{A}_-$, which acts on $\mathcal{H}_-$, are given by

$$(u_a)_{ij} = \frac{1}{m_0}\begin{pmatrix} \varphi_a & 0 & 0 \\ \psi_{a\alpha} & \varphi_a & 0 \\ F_a & -\psi_a^{\alpha} & \varphi_a \end{pmatrix} \in \mathcal{A}_+, \tag{35}$$

$$(\bar{u}_b)_{\bar{i}\bar{j}} = \frac{1}{m_0}\begin{pmatrix} \varphi_b^* & 0 & 0 \\ \bar{\psi}_b^{\dot{\alpha}} & \varphi_b^* & 0 \\ F_b^* & -\bar{\psi}_{b\dot{\alpha}} & \varphi_b^* \end{pmatrix} \in \mathcal{A}_-, \tag{36}$$

where $\{\varphi_a(\varphi_b^*), \psi_{a\alpha}(\bar{\psi}_b^{\dot{\alpha}}), F_a(F_b^*)\}$ are chiral (antichiral) multiplets. Note that these multiplets are not related to the multiplets in the functional space in Eqs. (7) and (8). As we will discuss in the next section, u_a and $\bar{u}_a$ together with the Dirac operator will be the origin of the gauge supermultiplets, while the elements (7) and (8) of the functional space are the origin of the matter fields.

The total supersymmetric Dirac operator D is defined as follows:

$$D = D_M \otimes 1 + \Gamma_M \otimes D_F. \tag{37}$$

On the basis $(\Phi, \Phi^c)^T$, the operator D_M is given by

$$D_M = \begin{pmatrix} \mathcal{D}_M & 0 \\ 0 & \mathcal{J}_M \mathcal{D}_M \mathcal{J}_M^{-1} \end{pmatrix}, \tag{38}$$

and

$$\mathcal{D}_M = -i \begin{pmatrix} 0 & \bar{\mathcal{D}}_{i\bar{j}} \\ \mathcal{D}_{\bar{i}j} & 0 \end{pmatrix}, \tag{39}$$

where

$$\mathcal{D}_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & i\bar{\sigma}^\mu \partial_\mu & 0 \\ \Box & 0 & 0 \end{pmatrix}, \ \bar{\mathcal{D}}_{i\bar{j}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & i\sigma^\mu \partial_\mu & 0 \\ \Box & 0 & 0 \end{pmatrix}. \tag{40}$$

The supersymmetric invariant product in $\mathcal{H}_M$ is defined by

$$(\Phi', \Phi) = \int_M d^4x \Phi'^\dagger \Gamma_0 \Phi, \tag{41}$$

where Γ_0 is given by

$$\Gamma_0 = \begin{pmatrix} 0 & \Gamma_0 \\ \Gamma_0 & 0 \end{pmatrix}, \tag{42}$$

and

$$\Gamma_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{43}$$

On the basis (Q_a, Q_a^c), the Dirac operator on the finite space D_F is defined as follows:

$$D_F = \begin{pmatrix} \mathcal{D}_F & 0 \\ 0 & \mathcal{J}_F \mathcal{D}_F \mathcal{J}_F^{-1}, \end{pmatrix} \tag{44}$$

and

$$\mathcal{D}_F = \begin{pmatrix} 0 & m^\dagger \\ m & 0 \end{pmatrix}, \tag{45}$$

where m is the mass matrix with respect to the family index.

The above formalism was given in the framework of the Minkowskian signature in order to incorporate supersymmetry. Since models in NCG in the flat space-time are constructed in the Euclidean space-time, if we see its correspondence to the NCG formalism, we must transform the variables in Minkowskian space coordinates to Euclidean ones. The transformation is given by the Wick rotation as follows:

$$x^0 \to -ix^0. \tag{46}$$

The algebra of $SL(2, C)$ turns out to be the algebra of $SU(2) \otimes SU(2)$ under the rotation. The Weyl spinors that transform as $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$ of $SL(2, C)$ are to be replaced by $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$

representations of $SU(2) \otimes SU(2)$, respectively. The spinors that have appeared in $\mathcal{H}_M$ and $\mathcal{A}_M$ are replaced as follows:

$$\psi_{(+)\alpha} \to \rho_{(+)\alpha},\ \psi^{\alpha}_{(+)} \to \rho^{\alpha *}_{(+)}, \tag{47}$$

$$\bar{\psi}^{\dot{\alpha}}_{(-)} \to \rho^{\dot{\alpha}}_{(-)},\ \bar{\psi}_{(-)\dot{\alpha}} \to \rho^{*}_{(-)\dot{\alpha}}, \tag{48}$$

where spinors with indices α transform as $(\frac{1}{2}, 0)$ and those with indices $\dot{\alpha}$ transform as $(0, \frac{1}{2})$ of $SU(2) \otimes SU(2)$, respectively. The upper index is related to the complex conjugate of the lower index by $\rho^1 = \rho_2^*$, $\rho^2 = -\rho_1^*$, $\rho^{\dot{1}} = \rho_{\dot{2}}^*$, $\rho^{\dot{2}} = -\rho_{\dot{1}}^*$. The metric and Pauli matrices that have appeared in the Dirac operator are to be replaced by

$$g^{\mu\nu} = (-1, 1, 1, 1) \to \eta^{\mu\nu} = (1, 1, 1, 1) \tag{49}$$

$$\sigma_\mu \to \sigma_E^\mu = (\sigma^0, i\sigma^i),\ \bar{\sigma}^\mu \to \bar{\sigma}_E^\mu = (\sigma^0, -i\sigma^i). \tag{50}$$

Embedding these expressions (47–50), the triple is rewritten in the Euclidean signature. The basis of $\mathcal{H}_M$ is denoted by the same form as (7), (8) but now Ψ_+ and Ψ_- are given by

$$(\Psi_+)_i = (\varphi_+, \rho_{+\alpha}, F_+), \tag{51}$$

$$(\Psi_-)_{\bar{i}} = (\varphi_-^*, \rho_-^{\dot{\alpha}}, F_-^*). \tag{52}$$

The elements of $\mathcal{A}_M$ that correspond to (35) and (36) are now given by

$$(u_a)_{ij} = \frac{1}{m_0}\begin{pmatrix} \varphi_a & 0 & 0 \\ \rho_{a\alpha} & \varphi_a & 0 \\ F_a & -\rho_a^\alpha & \varphi_a \end{pmatrix} \in \mathcal{A}_+, \tag{53}$$

$$(\bar{u}_b)_{\bar{i}\bar{j}} = \frac{1}{m_0}\begin{pmatrix} \varphi_b^* & 0 & 0 \\ \rho_b^{\dot{\alpha}} & \varphi_b^* & 0 \\ F_b^* & -\rho_{b\dot{\alpha}} & \varphi_b^* \end{pmatrix} \in \mathcal{A}_-. \tag{54}$$

For the Dirac operator on the Minkowskian manifold in (39) and (40), we have

$$\mathcal{D}_M = -i\begin{pmatrix} 0 & \bar{\mathcal{D}}_E \\ \mathcal{D}_E & 0 \end{pmatrix}, \tag{55}$$

where

$$\mathcal{D}_{E\bar{i}j} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \bar{\sigma}_E^\mu \partial_\mu & 0 \\ \Box_E & 0 & 0 \end{pmatrix},\ \bar{\mathcal{D}}_{Ei\bar{j}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \sigma_E^\mu \partial_\mu & 0 \\ \Box_E & 0 & 0 \end{pmatrix}, \tag{56}$$

with

$$\Box_E = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_0^2 + \partial_i^2. \tag{57}$$

The invariant product under Euclidean supersymmetry transformation is given by the same form as (41), but Γ_0 should be replaced by

$$\Gamma_0 = \begin{pmatrix} \Gamma_0 & 0 \\ 0 & \Gamma_0 \end{pmatrix}. \tag{58}$$

Under these replacements with the Wick rotation, when we restrict the functional space $\mathcal{H}$ to its fermionic part $\mathcal{H}_0$, we recover the original spectral triple and the formalism of NCG.

3. Internal fluctuation and vector supermultiplet

In the supersymmetric counterpart of the NCG, the vector superfield is to be introduced as the internal fluctuation of the Dirac operator D:

$$D \to \tilde{D} = D + V + JVJ^{-1},\ V = \sum_a U_a'[D, U_a],\ U_a \in \mathcal{A}, \tag{59}$$

where $J = J_M \otimes J_F$. We assume that $\mathcal{A}_F$ is the algebra of $N \times N$ complex matrix functions for the space of Q^a. As the algebra $\mathcal{A}_M$ is a direct sum of $\mathcal{A}_+$ and $\mathcal{A}_-$, we need two sets of elements, Π_+ and Π_-:

$$\Pi_+ = \{u_a : a = 1, 2, \dots n\} \subset \mathcal{A}_+ \otimes \mathcal{A}_F \tag{60}$$

$$\Pi_- = \{\bar{u}_a : a = 1, 2, \dots n\} \subset \mathcal{A}_- \otimes \mathcal{A}_F, \tag{61}$$

where u_a and $\bar{u}_a$ are given in the matrix form of (35) and (36) in the space of Q^a. They are also $N \times N$ complex matrix functions that act on the internal degrees of freedom of $\mathcal{H}_F$.

On the other hand, we assume that the algebra $\mathcal{A}$ for the space of the antiparticles Q_c^a is that of the constant complex number c, so that its elements are proportional to the unit matrix in $\mathcal{A}_M$. Then the fluctuation induced by the term $c[\mathcal{D}_M, c]$ vanishes. But the term JVJ^{-1} carries the same non-vanishing fluctuation induced by $N \times N$ complex matrices in the space of Q^a to the space of Q_a^c.

Since the product of chiral (antichiral) supermultiplets is again the chiral (antichiral) supermultiplet, the elements of Π_+ (Π_-) are chosen such that the products of two or more $u_a's(\bar{u}_a s)$ do not belong to Π_+ (Π_-) any more.

We shall define the following scalar, spinor, and vector superfields as the bilinear form of the two component functions in $u_a \in \Pi_+$ and $\bar{u}_a \in \Pi_-$:

$$m_0^2 C = \sum_a c_a \varphi_a^* \varphi_a, \tag{62}$$

$$m_0^2 \chi_\alpha = -i\sqrt{2} \sum_a c_a \varphi_a^* \psi_{a\alpha}, \tag{63}$$

$$m_0^2 (M + iN) = -2i \sum_a c_a \varphi_a^* F_a, \tag{64}$$

$$m_0^2 A_\mu = -i \sum_a c_a \Big[\Big(\varphi_a^* \partial_\mu \varphi_a - \partial_\mu \varphi_a^* \varphi_a \Big) - i \bar{\psi}_{a\dot{\alpha}} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \psi_{a\alpha} \Big], \tag{65}$$

$$m_0^2 \lambda_\alpha = \sqrt{2} i \sum_a c_a \Big(F_a^* \psi_{a\alpha} - i \sigma_{\alpha\dot{\alpha}}^\mu \bar{\psi}_a^{\dot{\alpha}} \partial_\mu \varphi_a \Big), \tag{66}$$

$$m_0^2 D = \sum_a c_a \Big[2F_a^* F_a - 2(\partial^\mu \varphi_a^* \partial_\mu \varphi_a) + i \Big\{ \partial_\mu \bar{\psi}_{a\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \psi_{a\alpha} - \bar{\psi}_{a\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu \psi_{a\alpha} \Big\} \Big], \tag{67}$$

where c_a are real coefficients. Using (9) and (10), we can show that these fields have the transformation property of the vector supermultiplet expressed by

$$\delta_\xi C = i\xi^\alpha \chi_\alpha - i\bar{\xi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}, \tag{68}$$

$$\delta_\xi \chi_\alpha = -i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}\Big(-A_\mu + i\partial_\mu C\Big) + \xi_\alpha(M+iN), \tag{69}$$

$$\frac{1}{2}\delta_\xi(M+iN) = \bar{\xi}_{\dot{\alpha}}\Big(\bar{\lambda}^{\dot{\alpha}} + i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\chi_\alpha\Big), \tag{70}$$

$$\delta_\xi A^\mu = i\xi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} + i\bar{\xi}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\lambda_\alpha + \xi^\alpha\partial^\mu\chi_\alpha + \bar{\xi}_{\dot{\alpha}}\partial_\mu\bar{\chi}^{\dot{\alpha}}, \tag{71}$$

$$\delta_\xi\lambda_\alpha = \sigma^{\mu\nu\beta}_\alpha\xi_\beta\Big(\partial_\mu A_\nu - \partial_\nu A_\mu\Big) + i\xi_\alpha D, \tag{72}$$

$$\delta_\xi D = -\xi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\lambda_\alpha. \tag{73}$$

If we express these fields as the superfield, we have

$$\begin{aligned} V(x,\theta,\bar{\theta}) = {} & C + \theta^\alpha(i\chi_\alpha) + \bar{\theta}_{\dot{\alpha}}(-i\chi^{\dot{\alpha}}) + \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}(-A_\mu) \\ & + \theta\theta\left[\frac{i}{2}(M+iN)\right] + \bar{\theta}\bar{\theta}\left[-\frac{i}{2}(M-iN)\right] \\ & + \theta\theta\bar{\theta}_{\dot{\alpha}}\left[i\left(\bar{\lambda}^{\dot{\alpha}} + \frac{i}{2}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\chi_\alpha\right)\right] + \bar{\theta}\bar{\theta}\theta^\alpha\left[-i\left(\lambda_\alpha + \frac{i}{2}\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\chi}^{\dot{\alpha}}\right)\right] \\ & + \theta\theta\bar{\theta}\bar{\theta}\left(\frac{1}{2}D + \frac{1}{4}\Box C\right). \end{aligned} \tag{74}$$

When we define the vector superfield (62–67), there is an ambiguity due to the choice of the algebraic elements. In order to see this, we consider two arbitrary elements of the algebra given by $u_0 \in \mathcal{A}_+$ and $\bar{u}_0 \in \mathcal{A}_-$. It turns out that the following functions obtained by these elements obey the supersymmetry transformation of a vector supermultiplet:

$$m_0 C_0 = \varphi_0 + \varphi_0^*, \tag{75}$$

$$m_0\chi_{0\alpha} = -i\sqrt{2}\psi_{0\alpha}, \tag{76}$$

$$m_0(M_0 + iN_0) = -2iF_0, \tag{77}$$

$$m_0 A_{0\mu} = -i\partial_\mu(\varphi_0 - \varphi_0^*), \tag{78}$$

$$\lambda_{0\alpha} = 0, \tag{79}$$

$$D_0 = 0. \tag{80}$$

Then we can redefine C, χ_α, M, N such that

$$C \to C + C_0 = 0, \tag{81}$$

$$\chi_\alpha \to \chi_\alpha + \chi_{0\alpha} = 0, \tag{82}$$

$$M + iN \to (M + M_0) + i(N + N_0) = 0. \tag{83}$$

To choose C, χ_α, M, and N in the vector supermultiplet to be zero is called the Wess–Zumino gauge. This gauge is realized in (62–67) by the following condition:

$$\sum_a c_a \varphi_a^* \varphi_a = 0,$$
$$\sum_a c_a \varphi_a^* \psi_a^\alpha = 0, \tag{84}$$
$$\sum_a c_a \varphi_a^* F_a = 0.$$

Hereafter let us call Eq. (84) the Wess–Zumino condition.

Since u_a and $\bar{u}_a$ are $N \times N$ complex matrix functions, A_μ, D, λ_α are also $N \times N$ complex matrix functions and are parametrized by

$$A_\mu(x) = \sum_{l=0}^{N^2-1} A_\mu^l(x) \frac{T_l}{2}, \tag{85}$$

$$D(x) = \sum_{l=0}^{N^2-1} D^l(x) \frac{T_l}{2}, \tag{86}$$

$$\lambda_\alpha(x) = \sum_{l=0}^{N^2-1} \lambda_\alpha^l(x) \frac{T_l}{2}. \tag{87}$$

Here, T_l is the basis of generators that belong to the fundamental representation of the Lie algebra associated with Lie group $U(N)$, which are normalized as follows:

$$\mathrm{Tr}(T_a T_b) = 2\delta_{ab}. \tag{88}$$

Since $A_\mu(x)$ and $D(x)$ are Hermitian, $A_\mu^l(x)$ and $D^l(x)$ are real functions. On the other hand, $\lambda_\alpha^l(x)$ are complex functions.

The supersymmetric Dirac operator modified by the fluctuation is denoted by

$$\tilde{\mathcal{D}}_M = -i \begin{pmatrix} 0 & \tilde{\bar{\mathcal{D}}}_{i\bar{j}} \\ \tilde{\mathcal{D}}_{\bar{i}j} & 0 \end{pmatrix}. \tag{89}$$

We consider the fluctuation due to $u_a \in \Pi_+$ and $\bar{u}_a \in \Pi_-$. In the basis of (5), we take U_a, U'_a in (59) as follows:

$$U_a = \sqrt{-2c_a} \begin{pmatrix} u_a & 0 \\ 0 & 0 \end{pmatrix}, \quad U'_a = \sqrt{-2c_a} \begin{pmatrix} 0 & 0 \\ 0 & \bar{u}_a \end{pmatrix}. \tag{90}$$

Then, the contribution to $\tilde{\mathcal{D}}_{\bar{i}j}$ is given by the following form:

$$\begin{aligned} V_{\bar{i}j} &= -2 \sum_a c_a (\bar{u}_a)_{\bar{i}\bar{k}} [i\mathcal{D}_M, u_a]_{\bar{k}j} \\ &= -2 \sum_a c_a (\bar{u}_a)_{\bar{i}\bar{k}} \mathcal{D}_{\bar{k}l} (u_a)_{lj}, \end{aligned} \tag{91}$$

and when we take U_a, U'_a as follows:

$$U_a = \sqrt{2c_a} \begin{pmatrix} 0 & 0 \\ 0 & \bar{u}_a \end{pmatrix}, \quad U'_a = \sqrt{2c_a} \begin{pmatrix} u_a & 0 \\ 0 & 0 \end{pmatrix}, \tag{92}$$

the contribution to $\tilde{\bar{D}}_{i\bar{j}}$ is given by

$$\begin{aligned}\bar{V}_{i\bar{j}} &= 2\sum_a c_a(u_a)_{ik}[i\mathcal{D}_M, \bar{u}_a]_{k\bar{j}} \\ &= 2\sum_a c_a(u_a)_{ik}\bar{\mathcal{D}}_{k\bar{l}}(\bar{u}_a)_{\bar{l}\bar{j}}.\end{aligned} \tag{93}$$

We shall calculate in the Wess–Zumino gauge. Using the definition of the vector supermultiplet given by (62–67), we obtain the following result:

$$V_{\bar{i}j} = -\begin{pmatrix} 0 & 0 & 0 \\ i\sqrt{2}\bar{\lambda}^{\dot{\alpha}} & -\bar{\sigma}^{\mu\dot{\alpha}\alpha}A_\mu & 0 \\ D + i\partial^\mu A_\mu + 2iA_\mu\partial^\mu & i\sqrt{2}\lambda^\alpha & 0 \end{pmatrix}, \tag{94}$$

and

$$\bar{V}_{i\bar{j}} = \begin{pmatrix} 0 & 0 & 0 \\ -i\sqrt{2}\lambda_\alpha & \sigma^\mu_{\alpha\dot{\alpha}}A_\mu & 0 \\ D - i\partial^\mu A_\mu - 2iA_\mu\partial^\mu & -i\sqrt{2}\bar{\lambda}_{\dot{\alpha}} & 0 \end{pmatrix}. \tag{95}$$

There is an additional fluctuation due to $u_{ab} = u_a u_b \in \mathcal{A}_+$ and $\bar{u}_{ab} = \bar{u}_a\bar{u}_b \in \mathcal{A}_-$, where $a, b = 1, \ldots, n$. This fluctuation is not contained in the fluctuation due to $u_a \in \Pi_+$ and $\bar{u}_a \in \Pi_-$, since $u_{ab} \notin \Pi_+$ and $\bar{u}_{ab} \notin \Pi_-$. The component fields of u_{ab} are expressed by the matrix form of (35) and (36) and each field is given by

$$u_{ab} = \{\varphi_{ab}, \psi_{ab\alpha}, F_{ab}\}, \tag{96}$$

where

$$\varphi_{ab} = \frac{1}{m_0}\varphi_a\varphi_b, \tag{97}$$

$$\psi_{ab\alpha} = \frac{1}{m_0}(\psi_{a\alpha}\varphi_b + \varphi_a\psi_{b\alpha}), \tag{98}$$

$$F_{ab} = \frac{1}{m_0}(\varphi_a F_b + F_a\varphi_b - \psi_a^\alpha\psi_{b\alpha}). \tag{99}$$

The component fields of $\bar{u}_{ab}$ are the complex conjugate functions of (97–99).

It turns out that the gauge-covariant form of $\tilde{\mathcal{D}}_M$ is obtained by adding the following fluctuation due to u_{ab} and $\bar{u}_{ab}$:

$$\begin{aligned}V'_{\bar{i}j} &= 2\sum_{a,b} c_a c_b(\bar{u}_{ab})_{\bar{i}\bar{k}}[i\mathcal{D}_M, u_{ab}]_{\bar{k}j} \\ &= 2\sum_{a,b} c_a c_b(\bar{u}_{ab})_{\bar{i}\bar{k}}\mathcal{D}_{\bar{k}l}(u_{ab})_{lj}\end{aligned} \tag{100}$$

and

$$\begin{aligned}\bar{V}'_{i\bar{j}} &= 2\sum_{a,b} c_a c_b(u_{ab})_{ik}[i\mathcal{D}_M, \bar{u}_{ab}]_{k\bar{j}} \\ &= 2\sum_{a,b} c_a c_b(u_{ab})_{ik}\bar{\mathcal{D}}_{k\bar{l}}(\bar{u}_{ab})_{\bar{l}\bar{j}}.\end{aligned} \tag{101}$$

Taking into account the Wess–Zumino gauge condition given by (84), we obtain

$$V'_{\bar{3}1} = -\bar{V}'_{3\bar{1}} = -A_\mu A^\mu, \tag{102}$$

and other matrix elements turn out to be zero. The fluctuation due to higher-order products of u_a or $\bar{u}_a$ such as $u_{abc} = u_a u_b u_c$ or $\bar{u}_{abc} = \bar{u}_a \bar{u}_b \bar{u}_c$ vanishes due to the Wess–Zumino gauge condition. Thus the total fluctuation in the Wess–Zumino gauge amounts to

$$V^{\mathrm{WZ}}_{\bar{i}j} = V_{\bar{i}j} + V'_{\bar{i}j}, \tag{103}$$

$$\bar{V}^{\mathrm{WZ}}_{i\bar{j}} = \bar{V}_{i\bar{j}} + \bar{V}'_{i\bar{j}}, \tag{104}$$

and the Dirac operator with fluctuation denoted by (89) is finally given by

$$\begin{aligned}\tilde{\mathcal{D}}_{\bar{i}j} &= \mathcal{D}_{\bar{i}j} + V^{\mathrm{WZ}}_{\bar{i}j} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ -i\sqrt{2}\bar{\lambda}^{\dot{\alpha}} & i\bar{\sigma}^{\mu}\mathcal{D}_{\mu} & 0 \\ \mathcal{D}_{\mu}\mathcal{D}^{\mu} - D & -i\sqrt{2}\lambda^{\alpha} & 0 \end{pmatrix},\end{aligned} \tag{105}$$

and

$$\begin{aligned}\tilde{\bar{\mathcal{D}}}_{i\bar{j}} &= \bar{\mathcal{D}}_{i\bar{j}} + \bar{V}^{\mathrm{WZ}}_{i\bar{j}} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ -i\sqrt{2}\lambda_{\alpha} & i\sigma^{\mu}\mathcal{D}_{\mu} & 0 \\ \mathcal{D}_{\mu}\mathcal{D}^{\mu} + D & -i\sqrt{2}\bar{\lambda}_{\dot{\alpha}} & 0 \end{pmatrix},\end{aligned} \tag{106}$$

where $\mathcal{D}_{\mu}$ is the covariant derivative,

$$\mathcal{D}_{\mu} = \partial_{\mu} - iA_{\mu}. \tag{107}$$

As for the Dirac operator on the finite space, we assume that $\mathcal{D}_F$ in (45) has no internal degrees of freedom, so the fluctuation for it does not arise.

The modified total Dirac operator on the basis $\Psi \otimes Q^a$ is given by

$$\tilde{\mathcal{D}}_{\mathrm{tot}} = \tilde{\mathcal{D}}_M - i\gamma_M \otimes \mathcal{D}_F. \tag{108}$$

Let us see that the counterpart of the first term in Eq. (2), which is in our supersymmetric case the part of the spectral action for the matter particles and their superpartners, is given by the bilinear form of supersymmetric invariant product (41) with the total Dirac operator in (108):

$$\begin{aligned}I_{\mathrm{matter}} &= (\Psi_L + \Psi_R, i\mathcal{D}_{\mathrm{tot}}(\Psi_L + \Psi_R)) \\ &= (\Psi_L + \Psi_R, i\mathcal{D}_M(\Psi_L + \Psi_R)) + (\Psi_L + \Psi_R, \gamma_M \otimes \mathcal{D}_F(\Psi_L + \Psi_R)) \\ &= (\Psi_L, \tilde{\mathcal{D}}\Psi_L) + (\Psi_R, \tilde{\bar{\mathcal{D}}}\Psi_R) + (\Psi_L, im^{\dagger}\Psi_R) - (\Psi_R, im\Psi_L),\end{aligned} \tag{109}$$

where Ψ_L, Ψ_R are the left-handed and right-handed particle supermultiplets in (15–22):

$$\Psi^a_L = \left(\tilde{q}^a_L, q^a_{L\alpha}, F^a_L, 0^3\right), \tag{110}$$

$$\Psi^a_R = \left(0^3, \tilde{q}^a_R, q^{a\dot{\alpha}}_R, F^a_R\right). \tag{111}$$

Using the definition of the supersymmetric invariant product (41–43) and fluctuated Dirac operator (89), (105), and (106), the kinetic parts of the matter particles are expressed by

$$\begin{aligned} I_L &= (\Psi_L, \tilde{\mathcal{D}}\Psi_L) \\ &= \int_M d^4x \Big[-\mathcal{D}_\mu \tilde{q}_L^* \mathcal{D}^\mu \tilde{q}_L - i\bar{q}_{L\dot{\alpha}} \bar{\sigma}^{\mu\dot{\alpha}\alpha} \mathcal{D}_\mu q_{L\alpha} \\ &\quad - \sqrt{2} i g (\tilde{q}_L^* \lambda^\alpha q_{L\alpha} - \bar{q}_{L\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}} \tilde{q}_L) - g \tilde{q}_L^* D \tilde{q}_L + F_L^* F_L \Big], \end{aligned} \tag{112}$$

and

$$\begin{aligned} I_R &= (\Psi_R, \tilde{\tilde{\mathcal{D}}}\Psi_R) \\ &= \int_M d^4x \Big[-\mathcal{D}_\mu \tilde{q}_R^* \mathcal{D}^\mu \tilde{q}_R - i\bar{q}_{R\alpha} \sigma^\mu_{\alpha\dot{\alpha}} \mathcal{D}_\mu q_R^{\dot{\alpha}} \\ &\quad - \sqrt{2} i g (\tilde{q}_R^* \bar{\lambda}_{\dot{\alpha}} q_R^{\dot{\alpha}} - \bar{q}_R^\alpha \lambda_\alpha \tilde{q}_R) - g \tilde{q}_R^* D \tilde{q}_R + F_R^* F_R \Big], \end{aligned} \tag{113}$$

where g is a rescale factor to the vector superfields that we will introduce later.

As for the mass terms, i.e, the last two terms in (109), we redefine the phase of Ψ_L as $\Psi_L \to i\Psi_L$, then we have

$$\begin{aligned} I_{\text{mass}} &= (\Psi_R, m\Psi_L) + \text{h.c.} \\ &= \int_M d^4x [\tilde{q}_R^* m F_L + F_R^* m \tilde{q}_L - \bar{q}_R^\alpha m q_{L\alpha} + \text{h.c.}]. \end{aligned} \tag{114}$$

4. Spectral action principle and super Yang–Mills action

Let us start the final task to derive the super Yang–Mills theory following the supersymmetric version of the prescription for constructing NCG particle models. In our noncommutative geometric approach to supersymmetry, we show that the action for the vector supermultiplet will be obtained by the coefficients of heat kernel expansion of the elliptic operator P:

$$Tr_{L^2} f(P) \simeq \sum_{n \geq 0} c_n a_n(P), \tag{115}$$

where $f(x)$ is an auxiliary smooth function on a smooth compact Riemannian manifold without boundary of dimension 4 similar to the non-supersymmetric case. Since the contribution to P from the antiparticles is the same as that of the particles, we consider only the contribution from the particles. Then the elliptic operator P in our case is given by the square of the Wick-rotated Euclidean Dirac operator $\tilde{\mathcal{D}}_{\text{tot}}$ expressed in the same form as (108):

$$\tilde{\mathcal{D}}_{\text{tot}} = \tilde{\mathcal{D}}_M - i\gamma_M \otimes \mathcal{D}_F, \tag{116}$$

where $\tilde{\mathcal{D}}_M$ is obtained from (89), (105), and (106) but with the replacement of (49), (50). Note that, as for the case in which internal fluctuation to $\mathcal{D}_F$ exists, we will discuss this in our next paper [13].

The elliptic operator P is expanded into the following form:

$$P = -(\eta^{\mu\nu} \partial_\mu \partial_\nu + \mathbb{A}^\mu \partial_\mu + \mathbb{B}). \tag{117}$$

The heat kernel coefficients a_n in Eq. (115) are found in Ref. [14]. They vanish for odd n, and the first three a_n for even n in the flat space are given by

$$a_0(P) = \frac{1}{16\pi^2}\int_M d^4x \, \mathrm{tr}_V(\mathbb{I}), \tag{118}$$

$$a_2(P) = \frac{1}{16\pi^2}\int_M d^4x \, \mathrm{tr}_V(\mathbb{E}), \tag{119}$$

$$a_4(P) = \frac{1}{32\pi^2}\int_M d^4x \, \mathrm{tr}_V\left(\mathbb{E}^2 + \tfrac{1}{3}\mathbb{E}^{\mu}_{;\mu} + \tfrac{1}{6}\Omega_{\mu\nu}\Omega^{\mu\nu}\right), \tag{120}$$

where $\mathbb{E}$ and the bundle curvature $\Omega^{\mu\nu}$ in the flat space are defined as follows:

$$\mathbb{E} = \mathbb{B} - (\partial_\mu\omega^\mu + \omega_\mu\omega^\mu), \tag{121}$$

$$\Omega^{\mu\nu} = \partial^\mu\omega^\nu - \partial^\nu\omega^\mu + [\omega^\mu, \omega^\nu], \tag{122}$$

$$\omega^\mu = \tfrac{1}{2}\mathbb{A}^\mu. \tag{123}$$

The coefficients c_n in (115) depend on the functional form of $f(x)$. If $f(x)$ is flat near 0, it turns out that $c_{2k} = 0$ for $k \geq 3$ and the heat kernel expansion terminates at $n = 4$ [8].

In Eqs. (118–120), tr_V denotes the trace over the vector bundle V. As for the supersymmetric theory we consider here, sections of the vector bundle V are smooth functions bearing indices that correspond to internal and spin degrees of freedom of the chiral and antichiral supermultiplets. For the spin degrees of freedom, tr_V is the supertrace defined by

$$\begin{aligned}\mathrm{Str}\, O &= \sum_i \langle i|(-1)^{2s} O|i\rangle \\ &= \sum_b \langle b|O|b\rangle - \sum_f \langle f|O|f\rangle,\end{aligned} \tag{124}$$

where s is the spin angular momentum and the states $|b\rangle$ and $|f\rangle$ stand for bosonic and fermionic states, respectively. Being attached with spinor indices, a matrix M represented on the space of supermultiplets spanned by the basis $(\varphi(x), \psi_\alpha(x), F(x))$ is expressed by

$$M = \begin{pmatrix} M_{11} & M_{12}^{\beta} & M_{13} \\ M_{21\alpha} & M_{22\alpha}^{\ \ \ \beta} & M_{23\alpha} \\ M_{31} & M_{32}^{\beta} & M_{33} \end{pmatrix}. \tag{125}$$

The supertrace of M is given by

$$\mathrm{Str}\, M = \mathrm{tr}_V M_{11} - \mathrm{tr}_V M_{22\alpha}^{\ \ \ \alpha} + \mathrm{tr}_V M_{33}. \tag{126}$$

The minus sign of $\mathrm{tr} M_{22}$ is due to the supersymmetry. In order to consider the meaning of the sign, we rewrite the diagonal elements of M on the superspace spanned by the basis $(\varphi, \theta^\beta\psi_\beta, \theta\theta F)$ as follows:

$$\mathrm{diag}\, M = (M_{11}, \theta^\alpha M_{22\alpha}^{\ \ \ \beta}\frac{\partial}{\partial\theta^\beta}, M_{33}). \tag{127}$$

It is reasonable to assume that

$$\mathrm{tr}_V(\theta^\alpha M_{22\alpha}^{\ \ \ \beta}\frac{\partial}{\partial\theta^\beta}) = -\mathrm{tr}_V(M_{22\alpha}^{\ \ \ \beta}\frac{\partial}{\partial\theta^\beta}\theta^\alpha) = -\mathrm{tr}_V M_{22\alpha}^{\ \ \ \alpha}, \tag{128}$$

so that (126) is established. In the same way, the supertrace of a matrix $\bar{M}$ represented on the space of the anti-supermultiplet $(\varphi^*, \bar{\psi}^{\dot{\alpha}}, F^*)$ is given by

$$\mathrm{Str}\, \bar{M} = \bar{M}_{11} - \bar{M}_{22\dot{\alpha}}^{\ \ \ \dot{\alpha}} + \bar{M}_{33}. \tag{129}$$

Now let us calculate the spectral action for $\tilde{\mathcal{D}}_{\text{tot}}^2$, where $\tilde{\mathcal{D}}_{\text{tot}}$ is given by (116). In the contribution to the spectral action from $\tilde{\mathcal{D}}_{\text{tot}}^2$, the terms including $\mathcal{D}_F$ vanish since $\mathcal{D}_M$ anticommutes with γ_M and

$$\text{Str}\ \gamma_M^2 = \text{Str}\ 1 = 0. \tag{130}$$

Thus, we consider the following elliptic operator P in the Euclidean signature:

$$P = \tilde{\mathcal{D}}_M^2 = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix}. \tag{131}$$

By making use of the Wick-rotated expressions (105) and (106), $P_\pm$ amounts to

$$\begin{aligned} P_+ &= -\tilde{\bar{\tilde{\mathcal{D}}}}_E \tilde{\mathcal{D}}_E \\ &= -\begin{pmatrix} \mathcal{D}_\mu \mathcal{D}^\mu - D & -i\sqrt{2}\lambda^\beta & 0 \\ -i\sqrt{2}\sigma^\mu_{E\alpha\dot{\alpha}}\left((\mathcal{D}_\mu \bar{\lambda}^{\dot{\alpha}}) + \bar{\lambda}^{\dot{\alpha}}\mathcal{D}_\mu\right) & \mathcal{D}_\mu \mathcal{D}^\mu \delta_\alpha^\beta + i\sigma^{\mu\nu\beta}_{E\alpha} F_{\mu\nu} & -i\sqrt{2}\lambda_\alpha \\ -2\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} & -i\sqrt{2}\bar{\lambda}_{\dot{\alpha}}\bar{\sigma}_E^{\mu\dot{\alpha}\beta}\mathcal{D}_\mu & \mathcal{D}_\mu \mathcal{D}^\mu + D \end{pmatrix}, \end{aligned} \tag{132}$$

$$\begin{aligned} P_- &= -\tilde{\mathcal{D}}_E \tilde{\bar{\tilde{\mathcal{D}}}}_E \\ &= -\begin{pmatrix} \mathcal{D}_\mu \mathcal{D}^\mu + D & -i\sqrt{2}\bar{\lambda}_{\dot{\beta}} & 0 \\ -i\sqrt{2}\bar{\sigma}_E^{\mu\dot{\alpha}\alpha}\left((\mathcal{D}_\mu \lambda_\alpha) + \lambda_\alpha \mathcal{D}_\mu\right) & \mathcal{D}_\mu \mathcal{D}^\mu \delta^{\dot{\beta}}_{\dot{\alpha}} + i\bar{\sigma}_E{}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}} F_{\mu\nu} & -i\sqrt{2}\bar{\lambda}^{\dot{\alpha}} \\ -2\lambda^\alpha \lambda_\alpha & -i\sqrt{2}\lambda^\alpha \sigma^\mu_{E\alpha\dot{\beta}}\mathcal{D}_\mu & \mathcal{D}_\mu \mathcal{D}^\mu - D \end{pmatrix}, \end{aligned} \tag{133}$$

In (132) and (133), $\sigma_E^{\mu\nu}$ and $\bar{\sigma}_E^{\mu\nu}$ are defined by

$$\sigma_E^{\mu\nu} = (i\sigma^{0j}, -\sigma^{ij}), \tag{134}$$
$$\bar{\sigma}_E^{\mu\nu} = (i\bar{\sigma}^{0j}, -\bar{\sigma}^{ij}), \tag{135}$$

and

$$\sigma_\alpha^{\mu\nu\beta} = \frac{1}{4}\left(\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\beta} - \sigma^\nu_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}\right), \tag{136}$$
$$\bar{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}\left(\bar{\sigma}^{\mu\dot{\alpha}\beta}\sigma^\nu_{\beta\dot{\beta}} - \bar{\sigma}^{\nu\dot{\alpha}\beta}\sigma^\mu_{\beta\dot{\beta}}\right). \tag{137}$$

The field A_μ, λ_α, D are the $N \times N$ matrices as shown in Eqs. (85–87). They turn out to be the gauge, gaugino, and auxiliary fields. The covariant derivative on spinors, say, λ_α, is given by

$$\mathcal{D}_\mu \lambda_\alpha = \partial_\mu \lambda_\alpha - i[A_\mu, \lambda_\alpha], \tag{138}$$

and $F_{\mu\nu}$ is the field strength defined by

$$\begin{aligned} F_{\mu\nu} &= i[\mathcal{D}_\mu, \mathcal{D}_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]. \end{aligned} \tag{139}$$

We expand $P_\pm$ in the form given by Eq. (117). Using the formulae (121) and (123), we obtain the following expressions:

$$\mathbb{E}_+ = \mathbb{B}_+ - (\partial_\mu \omega_+^\mu + \omega_{+\mu}\omega_+^\mu) = \begin{pmatrix} -D & -i\sqrt{2}\lambda^\beta & 0 \\ \frac{-i}{\sqrt{2}}\sigma^\mu_{E\alpha\dot{\alpha}}(\mathcal{D}_\mu\bar{\lambda}^{\dot{\alpha}}) & i\sigma^{\mu\nu\beta}_{E\alpha}F_{\mu\nu} & -i\sqrt{2}\lambda_\alpha \\ -2\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} & \frac{i}{\sqrt{2}}(\mathcal{D}_\mu\bar{\lambda}_{\dot{\alpha}})\bar{\sigma}^{\mu\dot{\alpha}\beta}_E & D \end{pmatrix} \tag{140}$$

$$\mathbb{E}_- = \mathbb{B}_- - (\partial_\mu \omega_-^\mu + \omega_{-\mu}\omega_-^\mu) = \begin{pmatrix} D & -i\sqrt{2}\bar{\lambda}_{\dot{\beta}} & 0 \\ \frac{-i}{\sqrt{2}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}_E(\mathcal{D}_\mu\lambda_\alpha) & i\bar{\sigma}^{\mu\nu\dot{\alpha}}_{E\ \ \dot{\beta}}F_{\mu\nu} & -i\sqrt{2}\bar{\lambda}^{\dot{\alpha}} \\ -2\lambda^\alpha\lambda_\alpha & \frac{i}{\sqrt{2}}(\mathcal{D}_\mu\lambda^\alpha)\sigma^\mu_{E\alpha\dot{\beta}} & -D \end{pmatrix}. \tag{141}$$

The bundle curvature $\Omega_\pm^{\mu\nu}$ given by (122) amounts to

$$\Omega_+^{\mu\nu} = \begin{pmatrix} -iF^{\mu\nu} & 0 & 0 \\ -\frac{i}{\sqrt{2}}\Big[\sigma^\nu_{E\alpha\dot{\alpha}}(\mathcal{D}^\mu\bar{\lambda}^{\dot{\alpha}}) - \sigma^\mu_{E\alpha\dot{\alpha}}(\mathcal{D}^\nu\bar{\lambda}^{\dot{\alpha}})\Big] & -iF^{\mu\nu}\delta_\alpha^\beta & 0 \\ 0 & -\frac{i}{\sqrt{2}}\Big[(\mathcal{D}^\mu\bar{\lambda}_{\dot{\alpha}})\bar{\sigma}^{\nu\dot{\alpha}\beta}_E - (\mathcal{D}^\nu\bar{\lambda}_{\dot{\alpha}})\bar{\sigma}^{\mu\dot{\alpha}\beta}_E\Big] & -iF^{\mu\nu} \end{pmatrix} \tag{142}$$

$$\Omega_-^{\mu\nu} = \begin{pmatrix} -iF^{\mu\nu} & 0 & 0 \\ -\frac{i}{\sqrt{2}}\Big[\bar{\sigma}^{\nu\dot{\alpha}\alpha}_E(\mathcal{D}^\mu\lambda_\alpha) - \bar{\sigma}^{\mu\dot{\alpha}\alpha}_E(\mathcal{D}^\nu\lambda_\alpha)\Big] & -iF^{\mu\nu}\delta_{\dot{\alpha}}^{\dot{\beta}} & 0 \\ 0 & -\frac{i}{\sqrt{2}}\Big[(\mathcal{D}^\mu\lambda^\alpha)\sigma^\nu_{E\alpha\dot{\beta}} - (\mathcal{D}^\nu\lambda^\alpha)\sigma^\mu_{E\alpha\dot{\beta}}\Big] & -iF^{\mu\nu} \end{pmatrix}. \tag{143}$$

From (140) and (141) we have

$$\mathrm{Str}\,\mathbb{E}_+ = \mathrm{Tr}[-D] - [-\sigma^{\mu\nu\alpha}_{E\alpha}]i\mathrm{Tr}F_{\mu\nu} + \mathrm{Tr}D = 0, \tag{144}$$

$$\mathrm{Str}\,\mathbb{E}_- = 0, \tag{145}$$

since

$$\sigma^{\mu\nu\alpha}_\alpha = \frac{1}{4}\Big(\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\alpha} - \sigma^\nu_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\Big) = -\frac{1}{2}\Big(g^{\mu\nu} - g^{\mu\nu}\Big) = 0, \tag{146}$$

$$\bar{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\alpha}} = 0. \tag{147}$$

As for the square of $\mathbb{E}^2$, we have

$$\begin{aligned} \mathrm{Str}\,\mathbb{E}_+^2 &= \mathrm{Tr}\Big[D^2 - \lambda^\beta\sigma^\mu_{E\beta\dot{\beta}}(\mathcal{D}_\mu\bar{\lambda}^{\dot{\beta}})\Big] \\ &\quad - \mathrm{Tr}\Big[\sigma^\mu_{E\alpha\dot{\alpha}}(\mathcal{D}_\mu\bar{\lambda}^{\dot{\alpha}})\lambda^\alpha - \sigma^{\mu\nu\beta}_{E\alpha}\sigma^{\lambda\kappa\alpha}_{E\beta}F_{\mu\nu}F_{\lambda\kappa} + \lambda_\alpha(\mathcal{D}_\mu\bar{\lambda}_{\dot{\beta}})\bar{\sigma}^{\mu\dot{\beta}\alpha}_E\Big] \\ &\quad + \mathrm{Tr}\Big[-(\mathcal{D}_\mu\bar{\lambda}_{\dot{\alpha}})\bar{\sigma}^{\dot{\alpha}\beta}_E\lambda_\beta + D^2\Big] \\ &= \mathrm{Tr}\Big[2D^2 - 4\bar{\lambda}_{\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta}_E(\mathcal{D}_\mu\lambda_\beta) - F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\varepsilon^{\mu\nu\lambda\kappa}F_{\mu\nu}F_{\lambda\kappa}\Big], \end{aligned} \tag{148}$$

and

$$\mathrm{Str}\,\mathbb{E}_-^2 = \mathrm{Tr}[2D^2 - 4\bar{\lambda}_{\dot{\beta}}\bar{\sigma}_E^{\mu\dot{\beta}\beta}(\mathcal{D}_\mu\lambda_\beta) - F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\varepsilon^{\mu\nu\lambda\kappa}F_{\mu\nu}F_{\lambda\kappa}], \tag{149}$$

where Tr denotes the trace over $N \times N$ matrices of the internal degrees of freedom. Equations (148) and (149) give the following expression:

$$\mathrm{tr}_V(\mathbb{E}^2) = \mathrm{Str}\,\mathbb{E}_+^2 + \mathrm{Str}\,\mathbb{E}_-^2 \tag{150}$$

$$= 2\mathrm{Tr}[2D^2 - 4\bar{\lambda}_{\dot{\beta}}\bar{\sigma}_E^{\mu\dot{\beta}\beta}(\mathcal{D}_\mu\lambda_\beta) - F_{\mu\nu}F^{\mu\nu}]. \tag{151}$$

The supertrace of $\Omega_{\pm\mu\nu}\Omega_\pm^{\mu\nu}$ amounts to

$$\mathrm{Str}\,\Omega_{\pm\mu\nu}\Omega_\pm^{\mu\nu} = \mathrm{Tr}[-F_{\mu\nu}F^{\mu\nu}] - \mathrm{Tr}[-F_{\mu\nu}F^{\mu\nu}\mathbf{1}_2] + \mathrm{Tr}[-F_{\mu\nu}F^{\mu\nu}] = 0. \tag{152}$$

Let us calculate the heat kernel coefficients. From (118), we obtain

$$a_0 = 0, \tag{153}$$

since the number of freedom of the bosonic sector is equal to the number of freedom of the fermionic sector due to the supersymmetry, so that $\mathrm{Str}\,\mathbb{I} = 0$. Equation (153) indicates that the cosmological constant vanishes in the supersymmetric theory. From Eqs. (144) and (145), the coefficient a_2 also vanishes:

$$a_2 = 0. \tag{154}$$

Finally, Eqs. (151) and (152) give

$$a_4 = \frac{1}{16\pi^2}\int_M dx^2\mathrm{Tr}\left[2D^2 - 4\bar{\lambda}_{\dot{\beta}}\bar{\sigma}_E^{\mu\dot{\beta}\beta}(\mathcal{D}_\mu\lambda_\beta) - F_{\mu\nu}F^{\mu\nu}\right], \tag{155}$$

since $\mathrm{tr}_V(\mathbb{E}^\mu_{;\mu}) = 0$.

The Euclidean super Yang–Mills action I_E is now given by

$$I_E = \mathrm{Tr}_{L^2} f(\tilde{\mathcal{D}}_M^2) = f_4 a_4. \tag{156}$$

In order to obtain the physical action we change the signature back to the Minkowskian $\eta^{\mu\nu} \to g^{\mu\nu}$ with $\sigma_E^\mu \to i\sigma^\mu$ and rescale the vector supermultiplet as $\{A_\mu, \lambda_\alpha, D\} \to \{gA_\mu, g\lambda_\alpha, gD\}$, where g turns out to be the gauge coupling constant. After this procedure we have the following super Yang–Mills action:

$$I_{\mathrm{SYM}} = \int_M dx^2\mathrm{Tr}\left[-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2i\bar{\lambda}_{\dot{\beta}}\bar{\sigma}^{\mu\dot{\beta}\beta}(\mathcal{D}_\mu\lambda_\beta) + D^2\right], \tag{157}$$

where we have fixed the constant f_4 such that

$$\frac{f_4}{8\pi^2} = \frac{1}{g^2}. \tag{158}$$

At last, we have arrived at the goal.

5. Conclusions

In this paper, we have introduced the supersymmetric counterpart of the spectral triple of NCG on the finite space as well as on the manifold investigated in Ref. [9]. We obtain the total Dirac operator (37). A vector supermultiplet is introduced as the internal fluctuation of the supersymmetrically extended Dirac operator $\mathcal{D}_M$ defined on the manifold. The modified Dirac operator $\tilde{\mathcal{D}}_M$ due to the fluctuation turns out to be supersymmetric and gauge-covariant. When we consider the algebra of $N \times N$ complex matrices that act on the finite space, the fluctuation induces $U(N)$ gauge degrees of freedom.

Following the prescription of NCG, we calculated the spectral action using our generalized supersymmetric Dirac operator $\tilde{\mathcal{D}}_{\mathrm{tot}}$ given in (116). The parts of the action that include kinetic terms and mass terms of matter fields and their superpartners are obtained from the supersymmetric invariant bilinear form. We calculated the coefficients of heat kernel expansion of the squared Dirac operator $P = \tilde{\mathcal{D}}_{\mathrm{tot}}^2$. The terms including $\mathcal{D}_F$ did not contribute to them. We have found that the expansion coefficients for $\mathcal{D}_M^2$ successfully derived the super Yang–Mills action. As a result, we expressed the whole supersymmetric action for the matter supermultiplets and vector supermultiplets by the simple formula of the spectral action principle.

The method proposed in this paper to calculate the spectral action is applicable to the derivation of the supersymmetric standard model. In this model, the Higgs bosons and their superpartners are introduced as the fluctuation to the supersymmetric Dirac operator on the finite space. Detailed discussions and calculations on this subject will be given in a separate paper [13].

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